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# Characterization of the Stability Radius via Bifurcation Techniques

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## Abstract

Robustness of stability of linear time-invariant systems using the relationship between the structured complex stability radius and a parametrized algebraic Riccati equation is analysed. Our approach is based on the observation that the algebraic Riccati equation can be viewed as a bifurcation problem. It is proved that the stability radius is, under certain assumptions, associated with a turning point of the bifurcation problem given by the parametrized algebraic Riccati equation. As a by-product, the stability radius can be computed via path following. Some numerical examples are presented.

KEY WORDS: Stability radius, bifurcation, turning point, robust stability, linear state space systems, Riccati equation

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## 1. Introduction

Consider a linear, finite dimensional, exponentially stable dynamical system of the form

$$\dot{x}(t) = Ax(t), \quad t \in \mathbb{R}, \quad \sigma(A) \subset \mathbb{C}_- \quad (1.1)$$

where  $A \in \mathbb{C}^{n \times n}$ ,  $\sigma(A)$  denotes the spectrum of  $A$  and

$$\mathbb{C}_- := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}. \quad (1.2)$$

We are interested in the robustness of the stability if (1.1) is perturbed by some  $P \in \mathbb{C}^{n \times n}$  to

$$\dot{x}(t) = [A + P]x(t). \quad (1.3)$$

More generally, the question is, to what extent stability is preserved when the entries of the nominal generator  $A$  are subjected to additive parameter perturbations of the form  $BPC$  so that the perturbed system has the form

$$\dot{x}(t) = [A + BPC]x(t), \quad t \in \mathbb{R} \quad (1.4)$$

Here  $B \in \mathbb{C}^{n \times m}$ ,  $C \in \mathbb{C}^{p \times n}$  are fixed matrices defining the *structure* of the perturbations and  $P \in \mathbb{C}^{m \times p}$  is an *unknown* disturbance matrix.

In recent years, the problem of robustness has regained a prominent position in systems theory. In particular, robustness measures for perturbations of the above kind have been introduced and analysed extensively by van Loan [21], Qui and Davison [18], Martin and Hwer [14], Biernacki, Hwang and Bhattacharyya [1]. Hinrichsen and Pritchard [8], [9] introduced the following concept of the stability radius:

Find the maximal number  $r > 0$  so that (1.4) is asymptotically stable for all  $P \in \mathbb{C}^{m \times p}$  with  $\|P\| < r$ . This number

$$r_{\mathbb{C}}(A; B, C) := \inf \{ \|P\|; P \in \mathbb{C}^{m \times p}, \sigma(A + BPC) \not\subset \mathbb{C}_- \} \quad (1.5)$$

is called the (*structured*) *stability radius* of  $A$  with respect to the perturbation structure  $(B, C)$ . Here  $\|P\|$  denotes the operator norm of the linear map  $P : \mathbb{C}^p \rightarrow \mathbb{C}^m$  with respect to the Euclidian norms on  $\mathbb{C}^m, \mathbb{C}^p$ .

If  $B = C = I_n$ , then

$$r_{\mathbb{C}}(A) := r_{\mathbb{C}}(A; I_n, I_n) \quad (1.6)$$

measures the distance of  $A$  from the set of not asymptotically stable matrices in  $\mathbb{C}^{n \times n}$ . If in (1.5) all matrices involved are *real* matrices, in particular the perturbation  $P$ , then we speak of the *real* stability radius. Real perturbations are for certain cases of more interest than complex ones. However, the complex stability radius is a lower bound for the real one and, most of all, yields valuable information about the robustness of a system with respect to wider classes of perturbations (time-varying, nonlinear and/or dynamic), see [9]. In a recent note, Hinrichsen and Pritchard [11] have shown that for a general class of *real* nonlinear and dynamic perturbations the *complex* stability radius is the relevant number.

This concept has been developed over the last three years to a comprehensive robustness analysis of linear state space systems and polynomials; see [5, 7, 10, 17], to name but a few.

In the following, we will only consider *real* matrices  $A, B, C$ . Nevertheless, the perturbation  $P$  is allowed to be *complex*.

We use the close relationship between the structured stability radius  $r_{\mathbb{C}}(A; B, C)$  and the existence of certain solutions of the parametrized algebraic Riccati equation

$R(X, \varrho) = 0$  where

$$\begin{aligned} R : \mathcal{S} \times \mathbb{R} &\rightarrow \mathcal{S} \\ (X, \varrho) &\mapsto A^T X + X A - \varrho^2 C^T C - X B B^T X \end{aligned} \quad (1.7)$$

is a nonlinear map and

$$\mathcal{S} := \{X \in \mathbb{R}^{n \times n} \mid X^T = X\}, \quad (1.8)$$

see Hinrichsen and Pritchard [9].

In the present paper we explore this relationship and consider the parametrized algebraic Riccati equation  $R(X, \varrho) = 0$  as a bifurcation problem. We prove that the stability radius is given by a quadratic turning point of  $R(X, \varrho) = 0$  – under some generic assumptions on  $A, B$  and  $C$ . Hence, the parametrized algebraic Riccati equation turns out to be an interesting example for bifurcation problems with turning points. As a by-product, this sets us in a position to refer to well used bifurcation algorithms for the determination of turning points via path following.

We proceed as follows. In Section 2 some well known results on the structured stability radius are presented. In Section 3 we collect the basic definitions and results of the concept of quadratic turning points. Finally, in Section 4, our main result, the characterization of the structured stability radius as a quadratic turning point is proved. The numerical procedure is explained in Section 5. We give some examples and compare our method with those presented in Hinrichsen, Kelb and Linnemann [6] and Hinrichsen and Motscha [7].

## 2. The stability radius and the algebraic Riccati equation.

In this section, we collect some basic results on the structured stability radius. The proofs of the results can be found in [8], [9].

Throughout the paper we assume (1.1) and that  $(A, B, C)$  are *real* matrices of dimension  $n \times n$ ,  $n \times m$ ,  $p \times n$ , respectively.

**2.1 Proposition** If  $G(s) = C(sI_n - A)^{-1}B$  denotes the transfer matrix associated with the system  $(A, B, C)$  then

$$r_{\mathbb{C}}(A; B, C) = \begin{cases} [\max_{\omega \in \mathbb{R}} \|G(i\omega)\|]^{-1} & \text{if } G \not\equiv 0 \\ \infty & \text{if } G \equiv 0 \end{cases}$$

## 2.2 Proposition

$$r_{\mathbb{C}}(A) = \min_{\omega \in \mathbb{R}} s_n(i\omega I_n - A)$$

where  $s_n(F)$  denotes the smallest singular value of  $F \in \mathbb{C}^{n \times n}$ .

The relationships between the stability radius and the parametrized algebraic Riccati equation  $R(X, \varrho) = 0$ , see (1.7), is given in the following proposition.

## 2.3 Proposition

(i) If  $\varrho \in [0, r_{\mathbb{C}}(A; B, C))$ , then there exists a solution  $P_{\varrho} \in \mathcal{S}$  of  $R(X, \varrho) = 0$  such that  $P_{\varrho}$  is *stabilizing*, i.e.  $\sigma(A - BB^T P_{\varrho}) \subset \mathbb{C}_-$ .  $P_{\varrho}$  is unique among all symmetric solutions with this property.

(ii) If  $\varrho = r_{\mathbb{C}}(A; B, C)$ , then there exists some  $P_{\varrho} \in \mathcal{S}$  which satisfies  $R(P_{\varrho}, \varrho) = 0$  and

$$\sigma(A - BB^T P_{\varrho}) \subset \overline{\mathbb{C}_-}, \quad \sigma(A - BB^T P_{\varrho}) \cap i\mathbb{R} \neq \emptyset. \quad (2.1)$$

(iii) For  $\varrho > r_{\mathbb{C}}(A; B, C)$  there does not exist a symmetric solution  $X$  of  $R(X, \varrho) = 0$ .

It is proved in [9] for complex  $(A, B, C)$  that a *complex* hermitian solution  $P_{\varrho}$  with the properties of Proposition 2.3 exists. However, if  $A, B, C$  are real, then a careful analysis shows that  $P_{\varrho}$  is a limit of *real* symmetric matrices and thus *real*, cf. [5].

## 3. Turning points in parameter dependent nonlinear equations.

We present a short review of the bifurcation analysis of a *turning point* for one-parameter dependent nonlinear equations

$$g(x, \lambda) = 0, \quad (3.1)$$

where  $g : X \times \mathbb{R} \rightarrow X$  is smooth and  $X$  is an  $n$ -dimensional vector space. In the following sections, we shall consider  $R(X, \varrho)$  instead of  $g(x, \lambda)$ .

*Simple  $\lambda$ -singular points*  $(x_0, \lambda_0)$  of (3.1) are defined by the properties that

$$g^0 = g(x_0, \lambda_0) = 0 \quad (3.2)$$

and

$$\dim \ker(g_x^0) = 1, \quad (3.3)$$

where the one-dimensional kernel of  $g_x^0$  is spanned by  $\varphi_0$ , being normalized by

$$l\varphi_0 = 1 \quad (3.4)$$

with a certain linear functional  $l$  on  $X$ .

*Quadratic turning points*  $(x_0, \lambda_0)$  of (3.1) are simple  $\lambda$ -singular points for which additionally

$$g_\lambda^0 \notin \text{im}(g_x^0) \quad (3.5)$$

and

$$g_{xx}^0 \varphi_0 \varphi_0 \notin \text{im}(g_x^0). \quad (3.6)$$

The name *quadratic turning point* origins from the following geometric properties of the solution set of (3.1).

From (3.3) and (3.5) it follows that the  $n \times (n+1)$  functional matrix  $Dg(x_0, \lambda_0)$  has full rank  $n$ . Hence locally near  $(x_0, \lambda_0)$ , (3.1) describes a smooth 1-dimensional manifold

$$\mathcal{C} := \{(x(s), \lambda(s)) : s \in I\}, \quad (x(s_0), \lambda(s_0)) = (x_0, \lambda_0).$$

The tangent direction of  $\mathcal{C}$  at  $(x_0, \lambda_0)$  is given by the kernel of  $Dg(x_0, \lambda_0)$  spanned by  $(\varphi_0, 0)$ . From here we have  $\lambda'(s_0) = 0$ . Differentiating the identity  $g(x(s), \lambda(s)) \equiv 0$  two times at  $s = s_0$ , one obtains from (3.6) that

$$\lambda''(s_0) \neq 0.$$

Hence  $s \mapsto \lambda(s)$  has a strict (quadratic) minimum or maximum at  $s = s_0$  and  $\mathcal{C}$  looks near  $(x_0, \lambda_0)$  like in Figure 1.

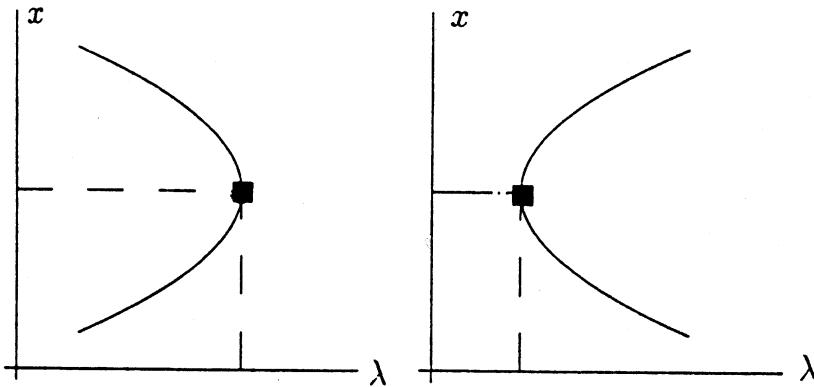


Figure 1: Turning points

An analysis of turning points can be found, for example, in Brezzi, Rappaz and Raviart [2]. Turning points are generic in one parameter problems (3.1). In corresponding dynamical systems  $\dot{x} = g(x, \lambda)$  they are related to the lost of stability of equilibria (nodes)  $x$  by variation of  $\lambda$  (*saddle node bifurcation*). The lost of stability is caused by a simple eigenvalue  $\mu(s)$  of the Jacobian  $A(s) := g_x(x(s), \lambda(s))$  crossing the imaginary axis of the complex plane at  $\mu = 0$ .

In elastic mechanics turning points are related to *snap through* phenomena, sometimes called *limit points*.

One of the first numerical aims in bifurcation problems was the computation of *turning points*. Instead of giving many references we only quote the survey article of Melhem and Rheinboldt [15]. Every method has to be embedded into a continuation method for following the path  $\mathcal{C}$  of solutions of (3.1). If no high precision is needed, very often this continuation is already sufficient since a quadratic turning point is easily detected by a sign change of  $\lambda'(s)$ .

One class of precise numerical methods for the computation of turning points is based on the following theorem (Moore-Spence [16], Spence-Werner [20]):

**3.1 Theorem** The conditions above for a quadratic turning point hold if and only if  $(x_0, \varphi_0, \lambda_0)$  is a regular root of

$$G : \begin{cases} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} & \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \\ (x, \varphi, \lambda) & \mapsto (g(x, \lambda), g_x(x, \lambda)\varphi, l\varphi - 1) \end{cases}$$

Now a Newton-like method with quadratic convergence can be performed to solve  $G = 0$ .

The same results hold true if the *extended* function  $G$  is replaced by

$$G(x, \lambda) := (g(x, \lambda), \Phi(x, \lambda)),$$

where  $\Phi(x, \lambda) = 0$  characterizes the singularity of  $g_x(x, \lambda)$ . An obvious choice (for other possibilities see Melhem and Rheinboldt [15]) is

$$\Phi(x, \lambda) := \det g_x(x, \lambda)$$

which we use for our computations in Section 5.

#### 4. Characterization of the stability radius by a turning point of the algebraic Riccati equation.

In this section, we apply the general results of Section 3 to the one-parameter dependent nonlinear equation (see (1.7), (1.8))

$$R(X, \varrho) = A^T X + X A - \varrho^2 C^T C - X B B^T X = 0.$$

$R$  is infinitely many times Fréchet differentiable. Differentiation yields

$$R_\varrho(X, \mu) = -2\mu C^T C \quad (4.1)$$

$$R_{XX}(X, \mu) H H = -2 H B B^T H \quad (4.2)$$

$$R_X(X, \mu) H = H(A - B B^T X) + (A - B B^T X)^T H \quad (4.3)$$

where  $H \in \mathcal{S}$  is arbitrary.

Consider the linear map  $\phi := R_X(P_\varrho, \varrho)$ ,

$$\begin{aligned} \phi : \mathcal{S} &\rightarrow \mathcal{S}, \\ H &\mapsto H A_\varrho + A_\varrho^T H \end{aligned} \quad (4.4)$$

where

$$A_\varrho := A - B B^T P_\varrho \quad (4.5)$$

and  $P_\varrho$  is a solution of  $R(X, \varrho) = 0$ .

Using the fact that the matrix equation  $A X = X B$  has only the trivial solution  $X = 0$  if  $\varrho(A) \cap \varrho(B) = \emptyset$  (Gantmacher [4]),  $\phi$  is an isomorphism if  $\varrho(A_\varrho) \subset \mathbb{C}_-$ . By Proposition 2.3 and the Implicit Function Theorem it follows that

- there is a unique branch  $\mathcal{C}_S := \{(P_\varrho, \varrho) : 0 \leq \varrho < r_{\mathbb{C}}(A; B, C)\}$  of solutions of  $R(X, \varrho) = 0$ , depending smoothly on  $\varrho$ .
- $\mathcal{C}_S$  cannot be continued for  $\varrho > r_{\mathbb{C}} := r_{\mathbb{C}}(A; B, C)$ . It "ends" at  $(P_{r_{\mathbb{C}}}, r_{\mathbb{C}})$ , where  $\phi = R_X(P_{r_{\mathbb{C}}}, r_{\mathbb{C}})$  has to be singular.

Hence the question arises how the solution set of  $R(X, \varrho) = 0$  looks like in a neighbourhood of  $(P_{r_{\mathbb{C}}}, r_{\mathbb{C}})$ . Our aim is the characterization of  $(P_{r_{\mathbb{C}}}, r_{\mathbb{C}})$  as a quadratic turning point by (3.3), (3.5) and (3.6). These conditions read in our setup as follows

$$\dim \ker \phi = 1, \quad (4.6)$$



$$C^T C \ni \operatorname{im} \phi \quad (4.7)$$

$$H_0 B B^T H_0 \ni \operatorname{im} \phi \quad \text{where } H_0 \in \ker \phi, \quad H_0 \neq 0. \quad (4.8)$$

The singularity of  $\phi := R_X(P_{r_{\mathbb{C}}}, r_{\mathbb{C}})$  is due to the fact that  $\sigma(A_{r_{\mathbb{C}}}) \subset \overline{\mathbb{C}}_-$  has eigenvalues in common with the imaginary axis. The next proposition shows that the number and the multiplicity of these (pairs of conjugate complex) eigenvalues essentially determine the dimension of  $\ker \phi$ . Hence (4.6) will be met if, for  $\varrho = r_{\mathbb{C}}$ ,  $\sigma(A_{\varrho})$  reaches the imaginary axis either at  $\mu = 0$  or at  $\mu = \pm i\omega$ ,  $\omega \neq 0$ , where  $\mu$  is algebraically simple (cf. Theorem 4.4).

**4.1 Proposition** Let  $A_{\varrho} \in \mathbb{R}^{n \times n}$ . Suppose  $\sigma(A_{\varrho}) \subset \overline{\mathbb{C}}_-$  and intersects the imaginary axis in semisimple eigenvalues  $\pm i\omega_j$ ,  $j \in \underline{r}$ . Counting the eigenvalues including their multiplicities and assuming

$$\omega_1, \dots, \omega_s > 0, \quad \omega_{s+1} = \dots = \omega_r = 0,$$

choose a system of  $r$  linearly independent eigenvectors  $u_j$  of  $A_{\varrho}$ , respectively  $v_j$  of  $A_{\varrho}^T$  such that

$$\begin{aligned} u_j, v_j &\in \mathbb{C}^n, \quad A_{\varrho} u_j = i\omega_j u_j, \quad A_{\varrho}^T v_j = i\omega_j v_j, & j = 1, \dots, s \\ u_j, v_j &\in \mathbb{R}^n, \quad A_{\varrho} u_j = 0, \quad A_{\varrho}^T v_j = 0, & j = s+1, \dots, r. \end{aligned} \quad (4.9)$$

Then

$$\ker \phi = \langle \operatorname{Re}(v_1 v_1^*), \dots, \operatorname{Re}(v_r v_r^*) \rangle_{\mathbb{R}} \quad (4.10)$$

and for every  $X \in \mathcal{S}$

$$X \in \operatorname{im} \phi \iff u_j^* X u_j = 0 \quad \text{for all } j \in \underline{r} \quad (4.11)$$

**Proof:** We consider the map

$$\tilde{\varphi} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}, \quad H \mapsto H A_{\varrho} + A_{\varrho}^T H.$$

It is easily calculated that

$$v_1 v_1^*, \dots, v_s v_s^*, \bar{v}_1 \bar{v}_1^*, \dots, \bar{v}_s \bar{v}_s^*, v_{s+1} v_{s+1}^*, \dots, v_r v_r^* \in \ker \tilde{\varphi}.$$

It follows from (4.9) and Gantmacher p. 219 [4] that

$$\dim_{\mathbb{C}} \ker \tilde{\varphi} = 2s + r - s = r + s$$

The linear independency of  $v_1, \dots, v_r$  implies

$$\ker \tilde{\varphi} = \langle v_1 v_1^*, \dots, v_s v_s^*, \bar{v}_1 \bar{v}_1^*, \dots, \bar{v}_s \bar{v}_s^*, v_{s+1} v_{s+1}^*, \dots, v_r v_r^* \rangle_{\mathbb{C}}.$$

Since  $\operatorname{Re}(v_j v_j^*) = \operatorname{Re}(\bar{v}_j \bar{v}_j^*) \in \mathcal{S}$  and  $\operatorname{Im}(v_j v_j^*) = \overline{\operatorname{Im}(\bar{v}_j \bar{v}_j^*)}$  are skew-symmetric it is easily verified that (4.10) holds true.

The direction " $\Rightarrow$ " in (4.11) is obvious. To prove " $\Leftarrow$ " define

$$\ell : \mathcal{S} \rightarrow \mathbb{R}^r, \quad X \mapsto (u_1^* X u_1, \dots, u_r^* X u_r)^T$$

Since  $\operatorname{im} \phi \subset \ker \ell$ , it remains to prove the inequality

$$r = \dim_{\mathbb{R}} \ker \phi = \operatorname{codim}_{\mathbb{R}} \operatorname{im} \phi \leq \operatorname{codim}_{\mathbb{R}} \ker \ell = \dim_{\mathbb{R}} \operatorname{im} \ell.$$

$u_1, \dots, u_r$  are linearly independent. Hence  $\dim_{\mathbb{R}} \operatorname{im} \ell = r$  and the proof is complete.  $\square$

Proposition 4.1 leads to system theoretic characterization of the conditions (4.7) and (4.8). For this we need the following definition.

**4.2 Definition** Consider the linear system

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\}, \quad t \in \mathbb{R} \quad (4.12)$$

denoted by  $(A, B, C)$  where  $A, B, C$  are complex matrices of formats  $n \times n, n \times m$ , resp.  $p \times n$ . The system (4.12) is called *detectable*, if for each solution  $x(\cdot)$  of the homogeneous system  $\dot{x}(t) = Ax(t)$  with  $Cx(\cdot) \equiv 0$  it follows that  $\lim_{t \rightarrow \infty} x(t) = 0$ . The system (4.12) is called *stabilizable*, if for every  $x_0 \in \mathbb{C}^n$  there exists a piecewise continuous function  $u(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{C}^m$  such that the solution  $x(\cdot)$  of  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $x(0) = x_0$ , satisfies  $\lim_{t \rightarrow \infty} x(t) = 0$ .

**4.3 Proposition** Under the assumptions of Proposition 4.1 we have

- (i)  $C^T C \notin \operatorname{im} \phi \iff (A_\theta, B, C)$  is detectable
- (ii)  $H_j B B^T H_j \notin \operatorname{im} \phi$  for all  $j \in \underline{r} \iff (A_\theta, B, C)$  is stabilizable  
where  $H_j := \operatorname{Re}(v_j v_j^*)$ ,  $v_j$  defined in (4.9).

**Proof:** (i) follows from (4.11) and the fact that  $(A_\theta, B, C)$  is detectable if and only if  $Cu_j \neq 0$  for all  $j \in \underline{r}$  (see e.g. [12]).

(ii) The following fact is well-known (see [12])

$$(A_\theta, B, C) \text{ is stabilizable} \iff B^T v_j \neq 0 \text{ for all } j \in \underline{r} \quad (4.13)$$

If  $\omega_j = 0$ , then it is easily shown that

$$H_j B B^T H_j \notin \text{im } \phi \iff B^T v_j \neq 0 \tag{4.14}$$

If  $\omega_j \neq 0$ , we proceed in several steps. Put

$$v_j = a + ib, \quad u_j = c + id, \quad a, b, c, d \in \mathbb{R}^n.$$

Then  $\det \begin{bmatrix} a^T c & a^T d \\ b^T c & b^T d \end{bmatrix} \neq 0$ , and therefore

$$B^T H_j u_j \neq 0 \iff [B^T a, B^T b] \begin{bmatrix} a^T c & a^T d \\ b^T c & b^T d \end{bmatrix} \neq 0 \iff B^T v_j \neq 0 \tag{4.15}$$

Since  $\text{Re}(v_j v_j^*) u_k = 0$  for all  $j \neq k$ , (4.11) implies

$$B^T H_j u_j \neq 0 \iff H_j B B^T H_j \notin \text{im } \phi \tag{4.16}$$

Now (ii) follows from (4.13) – (4.16). □

The following main result of the present paper is a consequence of Proposition 4.1, 4.3, and (4.6) – (4.8).

**4.4 Theorem**     Let  $r_{\mathbb{C}} := r_{\mathbb{C}}(A; B, C)$  and  $P_{r_{\mathbb{C}}} \in \mathcal{S}$  be a solution of  $R(X, r_{\mathbb{C}}) = 0$  so that  $A_{r_{\mathbb{C}}}$  satisfies (2.1).

Suppose  $(A_{r_{\mathbb{C}}}, B, C)$  is stabilizable and detectable and

$$\begin{aligned} &\text{either } \sigma(A_{\varrho}, r_{\mathbb{C}}) \text{ intersects the imaginary axis in } \mu = 0 \\ &\text{or in } \mu = \pm i\omega, \quad \omega > 0 \text{ with algebraically simple } \mu. \end{aligned} \tag{4.17}$$

Then a quadratic turning point of the parametrized Riccati equation  $R(X, \varrho) = 0$  occurs at  $\varrho = r_{\mathbb{C}}$ .

### 4.5 Remark

- (i) If the unstructured stability radius is considered, then  $(A_{r_{\mathbb{C}}}, I, I)$  is of course stabilizable and detectable.
- (ii) We are not able to present a sufficient condition in terms of  $A, B, C$  for the stabilizability and detectability of  $(A_{\varrho}, B, C)$ . However, these assumptions are generically satisfied within the space  $\mathbb{R}^{n^2+nm+np}$  (see [22], p. 44).
- (iii) The assumption (4.17) is of course more restrictive ( $r = 1$ ) than the corresponding assumptions about  $\sigma(A_{\varrho})$  in Proposition 4.1. But in all our

computations we met (4.17) which is apparently generic for arbitrary families of real matrices  $A_\varrho$ .

Nevertheless, it is an interesting open question whether the geometry of the solution set of  $R(X, \varrho) = 0$  near  $(P_{r_c}, r_c)$  can be characterized also in situations where (4.17) fails, for example if  $r > 1$  in Proposition 4.1 or if defective eigenvalues are allowed.

Proposition 4.1. shows that we have to handle *multiple* singular points if  $r > 1$ . Our conjecture is that other branches of  $R(X, \varrho) = 0$  with turning points will coalesce in a *generalized* turning point  $(P_{r_c}, r_c)$ .

But observe that the Newton method we will now present, postulates (4.17) to preserve quadratic convergence at a regular root.

## 5. Numerical procedure and examples.

Our algorithm for the computation of the stability radius  $r_c(A; B, C)$  based on Theorem 4.4 is as follows.

First we reformulate  $R(X, \varrho) = 0$  as a nonlinear system of equations with  $N := \frac{n(n+1)}{2}$  equations (the matrix  $X$  is symmetric) and  $N + 1$  unknowns (see the discussion above). Starting with  $\varrho = 0$  and  $X = 0$  we follow the branch of solutions  $(X(s), \varrho(s))$  of

$$R(X, \varrho) = A^T X + X A - \varrho^2 C^T C - X B B^T X = 0$$

into the direction of positive  $\varrho$  until we encounter a turning point at  $s = s_0$ .

Since in our problem  $\varrho'(0) > 0$ ,  $\varrho'(s_0) = 0$  and  $\varrho''(s_0) < 0$ , the turning point is easily detected, if the difference of the parameter values of two consecutive branch points becomes negative, that is if  $\varrho(s)$  starts to decrease.

Then we switch to the numerical solution of the following extended system

$$G(X, \varrho) = (R(X, \varrho), \det R_X(X, \varrho)) = 0$$

described in Section 3. We use a Newton method with the arithmetic mean of the last two consecutive branch points  $(X(s_i), \varrho(s_i))$ ,  $i = 1, 2, s_1 < s_2$ , as starting point. We stop the iteration if the norm of the defect of  $G$  is smaller than a given  $\varepsilon$ .

The path following code we use is essentially DERPARG of Kubiček (see Kubiček and Marek [13]) with an additional simple step size control. Here it is important that our aim is to reach the turning point of the path as fast as possible. We are not forced to keep the step size small in order to compute the path itself very accurate. On the other hand, the starting approximation for the turning point should not be too bad. Otherwise the Newton method fails to converge or converges too slowly.

Instead of DERPARG, we can use any available path following package, as for instance PITCON of Burkhart and Rheinboldt (see [19], where also general path following techniques are described).

As a test example we take Example 6.5 in Hinrichsen and Motscha [7]. Here

$$A = \begin{bmatrix} -0.201 & 0.755 & 0.351 & -0.075 & 0.033 \\ -0.149 & -0.696 & -0.160 & 0.110 & -0.048 \\ 0.081 & 0.004 & -0.189 & -0.003 & 0.001 \\ -0.173 & 0.802 & 0.251 & -0.804 & 0.056 \\ 0.092 & -0.467 & -0.127 & 0.075 & -1.162 \end{bmatrix}$$

For the unstructured stability radius we obtain

$$r_{\mathbb{C}}(A) = 0.11158200455.$$

This number coincides in eight digits with the number obtained in [7].

Setting

$$C := [1, 0, 0, 0, 0]$$

we obtain values for the structured stability radius according to the following table:

$B^T$	$r_{\mathbb{C}}(A; B, C)$
$(1, 0, 0, 0, 0)$	0.31038543595
$(0, 1, 0, 0, 0)$	0.26467891528
$(0, 0, 1, 0, 0)$	0.32408477578
$(0, 0, 0, 1, 0)$	5.00046308167
$(0, 0, 0, 0, 1)$	19.12826096370

The coincidence of our numbers with those in [7] are between five and seven digits. For quadratic turning points  $(X, r_{\mathbb{C}})$ , the accuracy of  $r_{\mathbb{C}}$  is asymptotically  $\varepsilon^2$ , if  $\varepsilon$  is the accuracy of  $X$ . Since the Newton method converges quadratically against  $(X, r_{\mathbb{C}})$ , we strongly believe that all the digits presented above are correct. But this is of course of minor practical meaning since a computation of the stability radius up to 12 digits seems to give no practical relevance.

For  $B$  given in the first line of the last table, the solution path of  $R(X, \varrho) = 0$  is visualized see Figure 2. We have followed this path beyond the turning point just for fun.

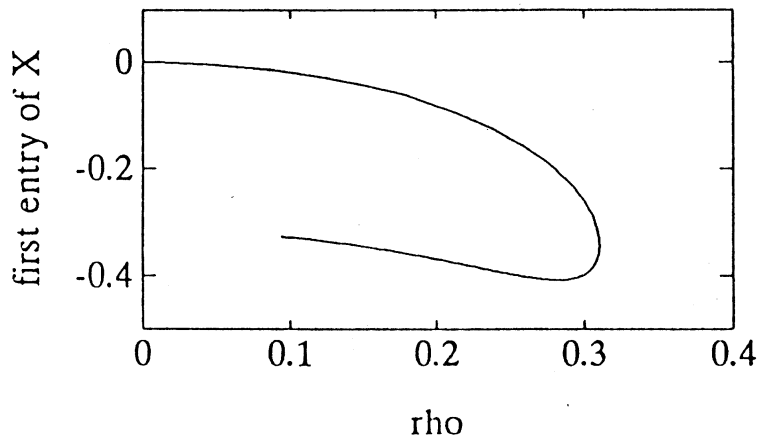


Figure 2: Solution path of  $R(X, \varrho) = 0$

The numerical results in Hinrichsen, Kelb and Linnemann [6] have also been recomputed by our procedure – without any difficulties.

The numerical methods used in Hinrichsen, Kelb and Linnemann [6] and Hinrichsen and Motscha [7] are also based on the algebraic Riccati equation. However, the essential theoretical fact being used is the following.  $\varrho$  is less than the complex stability radius  $r_{\mathbb{C}}(A; B, C)$  if and only if the Hamiltonian matrix

$$H_{\varrho} := \begin{bmatrix} A & -BB^T \\ \varrho CC^T & -A^T \end{bmatrix} \quad (5.1)$$

has no eigenvalues in common with the imaginary axis. Hence they suggest bisection like methods where in each iteration step *all* eigenvalues of the  $2n \times 2n$ -Hamiltonian matrix  $H_{\varrho}$  are computed. The speed of convergence of the iteration process is at least linear.

Concerning the effort, our method seems to be inferior at least for larger  $n$ . Comparing one iteration step, essentially a full  $2n \times 2n$ -Hamiltonian eigenvalue problem is solved in comparison with the solution of a linear system of dimension  $N+1$ , where  $N = (n+1)n/2$ . The advantage of our approach is the quadratic convergence in connection with a high accuracy. Moreover, we obtain the destabilizing matrix  $X = P_{\varrho}$  (cf. Proposition 2.3) without any further costs.

One should of course observe that we have solved the algebraic Riccati equation without taking into account any special structure of this equation. It is known, that for a given  $\varrho < r_{\mathbb{C}}(A; B, C)$  the stabilizing solution  $X = P_{\varrho}$  can also be

computed by means of the system of eigenvectors of the Hamiltonian  $H_\rho$  in (5.1). Moreover, there exist efficient symplectic QR like algorithms to solve this eigenvalue problem (see Bunse-Gerstner and Mehrmann [3]). But this works only for a *given*  $\rho$ , whereas the success (quadratic convergence) of our continuation approach is based on the fact that we do not distinguish between the parameter  $\rho$  and any of the elements of the matrix  $X$ . Nevertheless, we hope that in the future we can take into account also the special structure of the nonlinear problem  $R(X, \rho) = 0$ .

If the essential assumptions (4.17) in Theorem 4.4 fails (see Remark 4.5), our method will at least loose the quadratic convergence property. The latter can be guaranteed only for quadratic turning points as a consequence of Theorem 3.1.

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